

Minimum L^∞ Accelerations in Riemannian Manifolds

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Abstract

Riemannian cubics are critical points for the L^2 norm of acceleration of curves in Riemannian manifolds M . In the present paper the L^∞ norm replaces the L^2 norm, and a less direct argument is used to derive necessary conditions analogous to those in [17]. The necessary conditions are examined when M is a sphere or a bi-invariant Lie group.

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1. Introduction

Consider the task of moving a physical object between given configurations at given times, in a way that is *optimal* in some sense yet to be defined.

Example 1. Let $t_0 < t_1$ be given real numbers, and let $x_0, x_1 \in E^3$ be given, where E^m denotes Euclidean m -space. The task is to move a point of unit mass from x_0 to x_1 along a curve $x : [t_0, t_1] \rightarrow E^3$ so as to minimise the total kinetic energy

$$K(x) := \frac{1}{2} \int_{t_0}^{t_1} \|x^{(1)}(t)\|_E^2 dt$$

where $x^{(1)}$ denotes the first derivative of x (the velocity), and $\|\cdot\|_E$ is the Euclidean norm. There is a unique minimising path, namely the uniform-speed interpolating line segment.

Therefore, given real numbers $t_0 < t_1 < \dots < t_n$ and points $x_0, x_1, \dots, x_n \in E^3$, the unique minimal energy interpolating curve is a track-sum of n line segments, traversed at uniform speed within each $[t_{j-1}, t_j]$ where $j = 1, 2, \dots, n$.

In practice there may be problems if an object is moved in this way, due to instantaneous changes of momentum at junctions t_j for $j \neq 0, n$. So another measure of optimality might be used instead of K . A simple alternative is the mean squared norm of the acceleration

$$J_2(x) := \frac{1}{t_1 - t_0} \int_{t_0}^{t_n} \|x^{(2)}(t)\|^2 dt$$

where x varies over piecewise- C^2 interpolating curves. As seen in [8], the natural cubic spline is the unique minimiser of J_2 .

For objects more complicated than a point mass, similar considerations may hold.

Example 2. Suppose that we want to move a rigid body whose centre of mass is fixed through given configurations at times t_0, t_1, \dots, t_n . The trajectory becomes a curve $x : [t_0, t_n] \rightarrow M$, where M is the 3-dimensional rotation group $SO(3)$. The given configurations are points $x_j \in SO(3)$, and K is now given by

$$K(x) = \frac{1}{2} \int_{t_0}^{t_1} \|x^{(1)}\|_{x(t)}^2 dt$$

where $\|\cdot\|_z$ is the Riemannian norm at $z \in M$. As in the Euclidean case, a minimiser of K is a track-sum of geodesic arcs traversed at uniform speed within each $[t_{j-1}, t_j]$. Such a path might be impractical, for the same reasons as in Example 1. So, as before, we consider minimisers of

$$J_2(x) := \frac{1}{t_1 - t_n} \int_{t_0}^{t_n} \|\nabla_t x^{(1)}\|_{x(t)}^2 dt$$

where $\nabla_t x^{(1)}$ is the covariant acceleration of the interpolant $x : [t_0, t_n] \rightarrow M$, and is adapted to the Riemannian structure of M . Minimisers of J_2 , called Riemann cubic splines, are C^2 track-sums of Riemannian cubics defined over the t_{j-1}, t_j , with trivial covariant accelerations at t_0 and t_n .

Riemannian cubics are C^∞ curves that with more mathematical structure than either geodesics or cubic polynomials, and are given by the nonlinear ordinary differential equation [17]

$$\nabla_t^3 x^{(1)} + R(\nabla_t x^{(1)}, x^{(1)})x^{(1)} = \mathbf{0} \quad (1)$$

where R is the Riemannian curvature tensor field.

Riemannian cubics are reviewed in [23]. More details, including recent results, can be found in [18], [19], [20], [21], [22], [24], [26], [14], [6], [7], [12]. The present paper initiates an alternative line of enquiry, motivated by different engineering considerations.

Whereas J_2 measures the mean squared norm of the force applied to the moving object, it may sometimes be more important to minimise the *maximum* norm of applied force, namely

$$J_\infty(x) := \max\{\|\nabla_t x^{(1)}\|_{x(t)} : t \in [t_0, t_n]\}$$

where $x : [t_0, t_n] \rightarrow M$ varies over piecewise- C^2 interpolants. As seen in §10, in order to find minimising curves it suffices to take $n = 1$ and impose additional constraints on velocities at endpoints.

More precisely, given $t_0 < t_1$ together with $x_i \in M$ and $v_i \in TM_{x_i}$, a piecewise- C^2 curve $x : [t_0, t_1] \rightarrow M$ is F_{x_0, x_1, v_0, v_1} -feasible when $x(t_i) = x_i$ and $x^{(1)}(t_i) = v_i$ for $i = 0, 1$. A minimiser of J_∞ on F_{x_0, x_1, v_0, v_1} -feasible curves is said to be F_{x_0, x_1, v_0, v_1} -optimal. Our main result, Theorem 1, gives a necessary condition for a curve to be F_{x_0, x_1, v_0, v_1} -optimal. Examples are given in §7 and §8 for the cases where M is a sphere and a bi-invariant Lie group respectively.

§9 introduces the related notion of F_{x_0, x_1, v_0} -optimality, for which an analogous necessary condition is given by Theorem 2. Then Corollary 2 of §10 says that, if x minimises J_∞ over piecewise- C^2 interpolants of many points, then a least one track-summand of x satisfies the conclusions of Theorem 1 or Theorem 2.

2. Optimal Curves

Let M be a C^∞ manifold of dimension $m \geq 1$, with a C^∞ Riemannian metric $\langle \cdot, \cdot \rangle$. Let real numbers $t_0 < t_1$ be given. The derivative $x^{(1)}$ of a piecewise- C^2 curve $x : [t_0, t_1] \rightarrow M$ is piecewise- C^1 , and the Levi-Civita covariant acceleration $\nabla_t x^{(1)}$ is a piecewise-continuous field defined along x .

Definition 1. A piecewise- C^2 curve $x : [t_0, t_1] \rightarrow M$ is said to be feasible when $x(t_i) = x_i$ and $x^{(1)}(t_i) = v_i$ for $i = 0, 1$.

Let $F = F_{x_0, x_1, v_0, v_1} = F_{x_0, x_1, v_0, v_1, t_0, t_1}$ be the set of all feasible curves, and define $J : F \rightarrow \mathbb{R}$ by $J(x) := \max\{\|\nabla_t x^{(1)}\|_{x(t)} : t \in [t_0, t_1]\}$ where, for $w \in M$, $\|\cdot\|_w$ is the Riemannian norm on TM_w associated with the Riemannian inner product $\langle \cdot, \cdot \rangle_w$.

A minimizer of J is said to be *optimal*. Our main result is a necessary condition for optimality, in terms of

$$L(X) := \nabla_t^2 X + R(X, x^{(1)})x^{(1)}$$

where R denotes Riemannian curvature, and X a piecewise- C^2 field defined along x .

Theorem 1. *Let $x \in F$ be optimal. Then $\|\nabla_t x^{(1)}\|_{x(t)}$ is constant. For some piecewise- C^2 function $\Phi : [t_0, t_1] \rightarrow [0, \infty)$ with $\Phi^{-1}(0)$ discrete, we have*

$$L(\phi \nabla_t x^{(1)}) = \mathbf{0} \quad (2)$$

where $\phi(t) := \sqrt{\Phi(t)}$. Moreover x and ϕ are C^∞ except possibly where $\Phi(t) = 0$.

Example 3. *Let x be an optimal curve in an open convex subset M of Euclidean m -space E^m . Then (2) reads*

$$\frac{d^2}{dt^2}(\phi x^{(2)}) = \mathbf{0} \implies \phi(t)x^{(2)}(t) = A + Bt$$

where $A, B \in E^m$. We have $z\phi(t) = \|A + Bt\|$ where z is the constant length of $x^{(2)}$.

- If $z = 0$ then $A = B = \mathbf{0}$ and x is an affine line segment.
- If $A + Bt_2 = \mathbf{0}$ for some $t_2 \in [t_0, t_1]$ where $B \neq \mathbf{0}$ then

$$\phi(t) = \frac{|t - t_2|\|B\|}{a} \implies x^{(2)}(t) = z \operatorname{sign}(t - t_2) \frac{B}{\|B\|} \text{ for } t \neq t_2.$$

So x is a C^1 quadratic polynomial spline which is C^∞ except possibly at t_2 .

- If A and B are linearly independent then ϕ is never zero. By shifting $[t_0, t_1]$ we can suppose $\langle A, B \rangle = 0$. Setting $\alpha := \|A\|$ and $\beta := \|B\|$,

$$\begin{aligned} x^{(2)}(t) &= z \frac{A + Bt}{\sqrt{\alpha^2 + \beta^2 t^2}} \implies \\ x(t) &= z \frac{\beta t \log(\beta t + \sqrt{\alpha^2 + \beta^2 t^2}) - \sqrt{\alpha^2 + \beta^2 t^2}}{\beta^2} A + \\ &\quad z \frac{\alpha^2 \log(\beta t + \sqrt{\alpha^2 + \beta^2 t^2}) + \beta t \sqrt{\alpha^2 + \beta^2 t^2}}{2\beta^3} B + Ct + D \end{aligned}$$

where $C, D \in E^m$ are constants.

The conclusions of Example 3 also follow from the argument preceding the proof of Theorem 4.1 in §4 of [15]. However [15], uses linear methods from classical approximation theory, building on [9], [10], [11] among others. These methods are inapplicable in a more general Riemannian setting. For instance, in §7 where M is a sphere, or in §8 where M is a bi-invariant semisimple Lie group, there seems to be no alternative to Theorem 1.

The proof of Theorem 1 is less direct than the derivations of the analogous Euler-Lagrange equations for geodesics [13], elastic curves [27], and Riemannian cubics [17]. As with these other variational problems, we proceed by considering *extremals*, where an optimal curve is necessarily extremal, but we need two distinct notions of extremal.

In §3, so-called L^∞ -*extremals* are used to prove that $\|\nabla_t x^{(1)}(t)\|_{x(t)}$ is constant, and that optimal curves are *locally optimal*. After some preliminaries in §4, local optimality is used in §5 to reduce Theorem 1 to the special case where M is an open subset of \mathbb{R}^m with a nonstandard Riemannian metric. For this special case, constancy of $\|\nabla_t x^{(1)}(t)\|_{x(t)}$ is used to identify x with an extremal in the sense of *optimal control* for a control system with state variables in \mathbb{R}^{2m+1} and controls in the Euclidean unit sphere S^{m-1} . Then in §6 a local calculation based on the Pontryagin Maximum Principle [25] is used to prove Theorem 1 for the special case, and this completes the proof of Theorem 1 in general.

A more direct proof of Theorem 1 is possible for optimal curves whose image is contained in a single coordinate chart, based on the Pontryagin Principle for a slightly different optimal control problem where $\|\nabla_t x^{(1)}\|_{x(t)}$ is not assumed to be constant. Then §3 could be omitted, at the expense of generality. Another alternative might be to use the Pontryagin Principle for multiprocesses [5], but §3 seems less complicated.

It might be thought that an application of the geometric formulation of the Pontryagin Principle could replace the local calculations of §6, and perhaps do away with §3. For this to work, the space of controls would need to vary from point to point, which is not envisaged in [3], [2], [4]. Possibly this might be done in the framework of [1], but our local calculation has the advantage of being more elementary.

Having proved Theorem 1, we go on to investigate some consequences. In §7, M is taken to be the Euclidean unit m -sphere, and the differential equation (2) is rewritten as the system (4), (16), without covariant derivatives. Figure 1 in Example 4 shows a numerical solution for x .

In §8, M is taken to be a Lie group G with a bi-invariant Riemannian metric. The differential equation (2) reduces and integrates to a differential equation (18) in the

Lie algebra \mathcal{G} , in terms of a *Lie reductions* V, X_L . Another two conserved scalar quantities in terms of V, X_L are also noted, and it is shown that generically x can be recovered by quadrature from V, X_L . A class of so-called null optimal curves turn out to be the same as null Riemannian cubics [18], [20], [22] and Chapter 4 of [24]. The curious geometry of reductions of non-null optimal curves in $SO(3)$ is illustrated by Figures 2, 3, 4 of Example 5.

In §9, M is once more a general Riemannian manifold, and the notion of feasibility is relaxed, so that $x^{(1)}$ is specified only at one endpoint. Simple modifications to the proof of Theorem 1 result in an additional necessary condition, given in Theorem 2.

In §10 Theorems 1, 2 are adapted to a J_∞ analogue of the situation in Example 2, where the $x(t_i)$ are specified at various t_i and $x^{(1)}$ is unconstrained. As in Corollary 2, a necessary condition for such a feasible curve x to be optimal is that equation (2) should hold along at least one arc, with an additional conditions when the arc is terminal.

3. L^∞ -Extremals

A piecewise- C^2 field $W : [t_0, t_1] \rightarrow TM$ along $x \in F$ is *variational* when $W(t_i) = \nabla_t W|_{t=t_i} = \mathbf{0}$ for $i = 0, 1$. For W variational, a *variation* $h \mapsto x_h \in F$ is given by $x_h(t) := \exp_{x(t)}(hW(t))$, and

$$\left(\frac{\partial}{\partial h} x_h(t)\right)|_{h=0} = W(t).$$

For $x \in F$ set $S_x := \{s \in [t_0, t_1] : \|\nabla_t x^{(1)}(t)\|_{s=t} = J(x)\}$. Then S_x is nonempty and closed.

Definition 2. $x \in F$ is an L^∞ -extremal when, for any variational field W along x , we have $\langle L(W), \nabla_t x^{(1)} \rangle_{t=s_W} \geq 0$ for some $s_W \in S_x$.

Lemma 1. If x is optimal then x is an L^∞ -extremal.

Proof: From the definitions of x_h and L , we find, for any variational field W ,

$$\frac{\partial}{\partial h} (\|\nabla_t x_h^{(1)}\|^2)_{h=0} = 2\langle L(W), \nabla_t x^{(1)} \rangle.$$

Suppose $x \in F$ is not an L^∞ -extremal. Then, for some variational field W , $\langle L(W), \nabla_t x^{(1)} \rangle < 0$ at all $s \in S_x$. Therefore, and because S_x is compact, for h sufficiently small we

have $\|\nabla_t x_h^{(1)}\|_{t=s}^2 < J(x)^2$ for all $s \in S_x$. Then $\|\nabla_t x_h^{(1)}\|^2 < J(x)^2$ at all t in an open subset U of $[t_0, t_1]$ containing S_x . In particular, if $U = [t_0, t_1]$ then x is not optimal.

If $U \subset [t_0, t_1]$ then, because the nonempty set $[t_0, t_1] - U$ is compact, $\sup\{\|\nabla_t x_h^{(1)}\|^2 : t \notin U\}$ is $\|\nabla_t x_h^{(1)}\|_{t=t_*}^2$ for some $t_* \notin U$. Because $S_x \subset U$, $\|\nabla_t x_h^{(1)}\|_{t=t_*}^2 < J(x)^2$. So $\|\nabla_t x_h^{(1)}\|^2 < J(x)^2$ on $[t_0, t_1]$, and x is not optimal. This proves Lemma 1.

Lemma 2. *If x is an L^∞ -extremal then $\|\nabla_t x^{(1)}\|_{x(t)}$ is constant.*

Proof: If x is a geodesic the lemma holds trivially. For x not a geodesic we have $J(x) > 0$. Arguing by contradiction, suppose that the open set $U := [t_0, t_1] - S_x$ is nonempty. Then U contains an open interval of the form $(t_* - \delta, t_* + \delta)$ where $t_* \in [t_0, t_1]$ and $\delta > 0$. Define fields $\hat{W}_L : [t_0, t_* - \delta] \rightarrow TM$ along $x|_{[t_0, t_* - \delta]}$ and $\hat{W}_R : [t_* + \delta, t_1] \rightarrow TM$ along $x|_{[t_* + \delta, t_1]}$ by solving the linear initial and terminal value problems

$$\begin{aligned} L(\hat{W}_L) &= -\nabla_t x^{(1)} & \text{subject to} & \quad \hat{W}_L(t_0) = \nabla_t \hat{W}_L|_{t=t_0} = \mathbf{0} \\ L(\hat{W}_R) &= -\nabla_t x^{(1)} & \text{subject to} & \quad \hat{W}_R(t_1) = \nabla_t \hat{W}_R|_{t=t_1} = \mathbf{0}. \end{aligned}$$

Then \hat{W}_L and \hat{W}_R are the restrictions of a single variational field $W : [t_0, t_1] \rightarrow TM$ along x , for which $\langle L(W), \nabla_t x^{(1)} \rangle_{t=s} = -J(x)^2$ for all $s \in S_x$. Since x is an L^∞ -extremal we have a contradiction. This proves Lemma 2.

Incidentally, there is also a kind of converse.

Proposition 1. *Let $x \in F$ satisfy the necessary condition of Theorem 1. Then x is an L^∞ -extremal.*

Proof: Set $X(t) := \phi(t)\nabla_t x^{(1)}(t)$. If x is not an L^∞ -extremal then, for some variational field W ,

$$\langle \nabla_t^2 W + R(W, x^{(1)})x^{(1)}, X \rangle > 0 \quad \text{on } [t_0, t_1].$$

Using symmetries of the curvature tensor [13] (2.27), and equation (2), the left hand side is

$$\langle \nabla_t^2 W, X \rangle + \langle W, R(X, x^{(1)})x^{(1)} \rangle = \langle \nabla_t^2 W, X \rangle - \langle W, \nabla_t^2 X \rangle = r^{(1)}(t)$$

where $r : [t_0, t_1] \rightarrow \mathbb{R}$ is given by $r(t) := \langle \nabla_t W, X \rangle - \langle W, \nabla_t X \rangle$. So r is strictly increasing.

Because $W(t_i) = \nabla_t W|_{t=t_i} = \mathbf{0}$, $r(t_i) = 0$ for $i = 0, 1$. This is a contradiction because r is strictly increasing. So x is an L^∞ -extremal after all, and this proves Proposition 1.

Proposition 1 is not used for the proof of Theorem 1. Instead we need Lemma 3 below, which asserts that an optimal curve is also *locally optimal*. This would be trivial if $J(x)$ was the integral of a non-negative function of x and its derivatives, as with geodesics, elastic curves and Riemannian cubics. In the present situation a proof is needed.

Lemma 3. *Let $x \in F_{x_0, x_1, v_0, v_1, t_0, t_1}$ be optimal, and choose $s_0 < s_1$ in $[t_0, t_1]$. For $i = 0, 1$ set $y_i = x(s_i)$ and $w_i = x^{(1)}(s_i)$. Then the restriction $y = x|_{[s_0, s_1]}$ of x to $[s_0, s_1]$ is optimal, considered as a curve in $F_{y_0, y_1, w_0, w_1, s_0, s_1}$.*

Proof: By Lemma 2, $J(y) = J(x)$. If y is not optimal let $\bar{y} \in F_{y_0, y_1, w_0, w_1, s_0, s_1}$ be a curve with $\|\nabla_t \bar{y}^{(1)}\|_s < J(y)$ for all $s \in [s_0, s_1]$. The track-sum $\bar{x} : [t_0, t_1] \rightarrow M$ of $x|_{[t_0, s_0]}$, \bar{y} and $x|_{[s_1, t_1]}$ is piecewise- C^2 . Indeed $\bar{x} \in F_{x_0, x_1, v_0, v_1, t_0, t_1}$.

If $s_0 = t_0$ and $s_1 = t_1$ the Lemma holds trivially. So we can suppose without loss of generality that $t_2 \notin [s_0, s_1]$ for some $t_2 \in [t_0, t_1]$. So $J(\bar{x}) = \|\nabla_t \bar{x}^{(1)}\|_{t=t_2} \|x(t_2)\| = J(x)$. So \bar{x} is optimal. Since $\|\nabla_t \bar{y}^{(1)}\|_s < J(y)$ for all $s \in [s_0, s_1]$, $\|\nabla_t \bar{x}\|_{\bar{x}(t)}$ is nonconstant, and this contradicts Lemma 2. So y is optimal after all, and Lemma 3 is proved.

Given $s_0 = t_0 < s_1 < \dots < s_j < \dots < s_n = t_1$, any curve $x \in F$ is a track-sum of its restrictions to the subintervals $[s_{j-1}, s_j]$. If x is optimal then, by Lemma 3, so are the $x|_{[s_{j-1}, s_j]}$ with respect to the values and derivatives of x at s_{j-1} and s_j . So it suffices to prove Theorem 1 when x maps into a coordinate chart of M . In this case, M can be replaced by an open subset of \mathbb{R}^m equipped with some Riemannian metric $\langle \cdot, \cdot \rangle$. From now on suppose this has been done.

4. Local Geometry

Mainly to be clear in future on matters of notation, we briefly review some coordinate-based differential geometry, where M is taken as an open subset of \mathbb{R}^m . Readers may prefer to skip to Lemma 4, then §5 and onwards, referring back as necessary.

For any $v \in M \subseteq \mathbb{R}^m$ we have a possibly non-Euclidean inner product $\langle \cdot, \cdot \rangle_v$ on $\mathbb{R}^m = TM_v$. The associated norm is denoted by $\|\cdot\|_v$, and the dual $\tilde{\omega} \in \mathbb{R}^m$ of a linear form $\omega \in (\mathbb{R}^m)^*$ with respect to $\langle \cdot, \cdot \rangle_v$ is given by $\omega(w) = \langle \tilde{\omega}, w \rangle_v$ for any $w \in \mathbb{R}^m$.

Any vector $w \in \mathbb{R}^m$ can be written in the form $w^i e_i$, where summation over i is understood, where the $w^i \in \mathbb{R}$, and where e_1, e_2, \dots, e_m are the standard basis elements of \mathbb{R}^m . For any vector field W on M , $W = W^i e_i$ where the $W^i : M \rightarrow \mathbb{R}$. The

Levi-Civita covariant derivative $\nabla_Y W$ of W at $Y \in TM_v$ is the vector in $TM_v = \mathbb{R}^m$ given by

$$\nabla_Y W = \frac{\partial W}{\partial x_i} Y^i + \Gamma_v(Y, W)$$

where, for each $v \in M$, $\Gamma_v : TM_v \times TM_v \rightarrow TM_v$ is the symmetric bilinear form given by

$$\Gamma_v(Y, W) := \Gamma_{ij}^k(v) Y^i W^j e_k$$

and the *Christoffel symbols* Γ_{ij}^k are C^∞ functions for all $1 \leq i, j, k \leq m$. The Riemannian metric determines the Christoffel symbols according to formula (2.26) of [13], and thereby the *Riemannian curvature* $R_v(X, Y)Z$ which is trilinear in $X, Y, Z \in TM_v$ and given by

$$R(X, Y)Z = \left(\frac{\partial \Gamma_{pj}^i}{\partial x_k} - \frac{\partial \Gamma_{kj}^i}{\partial x_p} + \Gamma_{kq}^i \Gamma_{pj}^q - \Gamma_{pq}^i \Gamma_{kj}^q \right) X^k Y^p Z^j e_i \quad (3)$$

according to formula (2.20) of [13], where summation over i, j, k, p, q is understood.

Now, given $x \in F = F_{x_0, x_1, v_0, v_1}$, the velocity $y := x^{(1)}$ is a vector field along x , and the *covariant acceleration* of x is given by the formula

$$\nabla_t x^{(1)} := \nabla_{x^{(1)}(t)} y = y^{(1)}(t) + \Gamma(x(t))(y(t), y(t))$$

and an optimal curve is one for which the maximum Riemannian norm of the right hand side is minimised. By Lemmas 1, 2, it suffices to optimise over curves $x \in F$ for which $z := \|\nabla_t x^{(1)}\|_{x(t)}$ is constant. Since Theorem 1 holds trivially when $z = 0$, take $z > 0$.

Let $S := S^{m-1} \subset E^m$ be the unit sphere with respect to the *Euclidean* norm $\|\cdot\|_E$ and define $f : M \times S \rightarrow \mathbb{R}^m$ by

$$f(w, u) := \frac{u}{\|u\|_w}.$$

Lemma 4. *For any $u \in S$ and any $j = 1, 2, \dots, m$,*

$$\frac{\partial}{\partial w_j} f(w, u) = -\langle \Gamma(e_j, f(w, u)), f(w, u) \rangle_w f(w, u).$$

Proof of Lemma 4: Because the Levi-Civita covariant derivative is compatible with the Riemannian metric,

$$\|u\|_w \frac{\partial}{\partial w_j} \|u\|_w = \frac{1}{2} \frac{\partial}{\partial w_j} \langle u, u \rangle_w = \langle \Gamma(e_j, u), u \rangle_w \implies \frac{\partial}{\partial w_j} (\|u\|_w) = \langle \Gamma(e_j, f(w, u)), u \rangle_w$$

which proves Lemma 4.

5. Local Reduction to Optimal Control

For M open in \mathbb{R}^m , and any fixed $w \in M$, the assignment $u \mapsto f(w, u)$ is a diffeomorphism from S onto the unit sphere with respect to the *Riemannian* norm $\|\cdot\|_w$. So, for any nongeodesic $x \in F$ with $\|\nabla_t x^{(1)}(t)\|_{x(t)}$ constant, there are unique curves $y : [t_0, t_1] \rightarrow \mathbb{R}^m$, $u : [t_0, t_1] \rightarrow S$ and $z : [t_0, t_1] \rightarrow (0, \infty)$ satisfying

$$x^{(1)}(t) = y(t) \quad (4)$$

$$y^{(1)}(t) = zf(x(t), u(t)) - \Gamma(x(t))(y(t), y(t)) \quad (5)$$

$$z^{(1)}(t) = 0 \quad (6)$$

$$\text{with } x(t_0) = x_0, \quad x(t_1) = x_1, \quad \text{and } y(t_0) = v_0, \quad y(t_1) = v_1. \quad (7)$$

So an optimal curve x corresponds to a *control extremal*, namely an extremal $(x, y, z) : [t_0, t_1] \rightarrow M \times \mathbb{R}^m \times (0, \infty)$ with control $u : [t_0, t_1] \rightarrow S$ for the optimal control problem with dynamics (4), (5), (6), boundary conditions (7), minimising

$$(t_1 - t_0)z = \int_{t_0}^{t_1} z(t) dt.$$

The admissible controls are taken as the piecewise-continuous functions $u : [t_0, t_1] \rightarrow S$. This is equivalent to varying $x : [t_0, t_1]$ through piecewise- C^2 curves with $\|\nabla_t x^{(1)}\|_{x(t)}$ constant, while satisfying $x(t_i) = x_i$ and $x^{(1)}(t_i) = v_i$ for $i = 0, 1$.

As a control extremal, (x, y, z) and u satisfy the Pontryagin Maximum Principle [25], stated in terms of the Hamiltonian $H : \{0, 1\} \times (M \times \mathbb{R}^m \times (0, \infty)) \times (\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R})^* \times S \rightarrow \mathbb{R}$ where

$$H(\epsilon, x, y, z, \lambda, \mu, \nu, u) := -\epsilon z + \lambda(y) + \mu(zf(x, u) - \Gamma_x(y, y)).$$

The Pontryagin Principle concerns a number $\epsilon \in \{0, 1\}$ and a curve of costates $(\lambda, \mu, \nu) : [t_0, t_1] \rightarrow (\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R})^*$ associated with the control extremal (x, y, z) , satisfying

$$\lambda^{(1)} = \mu(z\langle \Gamma(e_j, \hat{u}), \hat{u} \rangle_x \hat{u} + d\Gamma_x(e_j)(y, y))e_j^* \quad (8)$$

$$\mu^{(1)} = -\lambda + 2\mu(\Gamma_x(e_j, y))e_j^* \quad (9)$$

$$\nu^{(1)} = \epsilon - \mu(\hat{u}) \quad (10)$$

for almost all $t \in [t_0, t_1]$. Here we use Lemma 4 to differentiate $\hat{u} := f(x, u)$, e_j^* is the dual of e_j , and we sum over $j = 1, 2, \dots, m$. Moreover, if $\epsilon = 0$ then (λ, μ, ν) is nowhere trivial.

Corresponding to the absence of conditions on $z(t_0)$ and $z(t_1)$ are transversality conditions

$$\nu(t_0) = \nu(t_1) = 0. \quad (11)$$

The Pontryagin Principle asserts that, for almost every $t \in [t_0, t_1]$, $u(t)$ is a maximiser of $H(\epsilon, x(t), y(t), z, \lambda(t), \mu(t), \nu(t), u)$, namely whenever $\mu(t) \neq \mathbf{0}$

$$u(t) = \frac{\tilde{\mu}(t)}{\|\tilde{\mu}(t)\|_E}.$$

6. Local Proof of Theorem 1

We continue to suppose that x is an optimal curve in an open subset M of \mathbb{R}^m .

Lemma 5. $\epsilon = 1$ and $\|\tilde{\mu}(t)\|_{x(t)}$ has mean 1.

Proof of Lemma 5 By (10) and transversality, $\int_{t_0}^{t_1} \|\tilde{\mu}(t)\|_{x(t)} dt = \epsilon(t_1 - t_0)$. So if $\epsilon = 0$ we must have μ identically $\mathbf{0}$. Then $\nu \equiv \mathbf{0}$ by (10) and transversality, and $\lambda \equiv \mathbf{0}$ by (9). But for $\epsilon = 0$, (λ, μ, ν) is nowhere trivial. So $\epsilon = 1$ after all, and $\int_{t_0}^{t_1} \|\tilde{\mu}(t)\|_{x(t)} dt = t_1 - t_0$. So Lemma 5 is proved.

Lemma 6. $L(\tilde{\mu})$ is identically $\mathbf{0}$.

Proof of Lemma 6: For $t \in [t_0, t_1]$, rewrite (8) and (9) as

$$\nabla_t \lambda = (\mu(z\langle \Gamma(e_j, \hat{u}), \hat{u} \rangle_x \hat{u} + d\Gamma_x(e_j)(y, y)) - \lambda(\nabla_t e_j))e_j^* \quad (12)$$

$$\nabla_t \mu = -\lambda + \mu(\nabla_t e_j)e_j^*. \quad (13)$$

Substituting for λ from (13) in (12),

$$\nabla_t \lambda = (\mu(z\langle \Gamma(e_j, \hat{u}), \hat{u} \rangle_x \hat{u} + d\Gamma_x(e_j)(y, y) - e_k^*(\nabla_t e_j)\nabla_t e_k) + \nabla_t(\mu)(\nabla_t e_j)e_j^*. \quad (14)$$

Differentiating (13) and substituting for $\nabla_t \lambda$ from (14), we find

$$\begin{aligned} \nabla_t^2 \mu &= -\nabla_t \lambda + (\nabla_t \mu)(\nabla_t e_j)e_j^* + \mu(\nabla_t^2 e_j)e_j^* + \mu(\nabla_t e_j)\nabla_t e_j^* = \\ &= -\mu(z\langle \Gamma(e_j, \hat{u}), \hat{u} \rangle_x \hat{u} + d\Gamma_x(e_j)(y, y) - \nabla_t^2 e_j)e_j^*. \end{aligned}$$

$$\text{Now } d\Gamma_x(e_j)(y, y) - \nabla_t^2 e_j = \frac{\partial \Gamma_{p,q}^k}{\partial x_j} y_p y_q e_k - \nabla_t(\Gamma_{pj}^k y_p e_k) =$$

$$\frac{\partial \Gamma_{p,q}^k}{\partial x_j} y_p y_q e_k - \frac{\partial \Gamma_{pj}^k}{\partial x_q} y_p y_q e_k - \Gamma_{pj}^k y_p^{(1)} e_k - \Gamma_{pj}^r \Gamma_{qr}^k y_p y_q e_k =$$

$$\frac{\partial \Gamma_{p,q}^k}{\partial x_j} y_p y_q e_k - \left(\frac{\partial \Gamma_{pj}^k}{\partial x_q} - \Gamma_{rj}^k \Gamma_{q,p}^r + \Gamma_{pj}^r \Gamma_{qr}^k \right) y_p y_q e_k - z \Gamma_{pj}^k \hat{u}_p e_k = R_{pjq}^k y_p y_q e_k - z \Gamma_{pj}^k \hat{u}_p e_k$$

by (5), (3). Because $\mu(\hat{u})\hat{u} = \|\tilde{\mu}\|_x \hat{u} = \tilde{\mu}$, we have $\mu(\langle \Gamma(e_j, \hat{u}), \hat{u} \rangle_x \hat{u} - \Gamma_{pj}^k \hat{u}_p e_k) =$

$$\langle \Gamma(e_j, \hat{u}), \hat{u} \rangle_x \|\tilde{\mu}\|_x - \mu(\Gamma(e_j, \hat{u})) = \langle \Gamma(e_j, \hat{u}), \tilde{\mu} \rangle_x - \mu(\Gamma(e_j, \hat{u})) = 0.$$

So $\nabla_t^2 \mu = -\mu(R_{pjq}^k y_p y_q e_k) e_j^*$, and

$$\nabla_t^2 \tilde{\mu} = -\langle \tilde{\mu}, R_{pjq}^k y_p y_q e_k \rangle_x e_j = -\langle \tilde{R}(e_j, y) y, \tilde{\mu} \rangle_x e_j = -\langle R(\tilde{\mu}, y) y, e_j \rangle_x e_j = -R(\tilde{\mu}, y) y$$

using symmetries of Riemannian curvature [13] (2.27). This proves Lemma 6.

So $\tilde{\mu}$ is a piecewise- C^2 field defined along the optimal curve x . Defining $\Phi : [t_0, t_1] \rightarrow [0, \infty)$ by

$$\Phi(t) := \|\tilde{\mu}(t)\|_{x(t)}^2,$$

we have $L(\sqrt{\Phi(t)} \nabla_t x^{(1)}) = L(z\tilde{\mu}) = zL(\tilde{\mu}) = \mathbf{0}$, since z is constant and by Lemma 6.

Lemma 7. $\Phi^{-1}(0)$ is a discrete subset of $[t_0, t_1]$.

Proof of Lemma 7: Let s_* be an accumulation point of $\Phi^{-1}(0)$. Then $\Phi(s_*) = \Phi^{(1)}(s_*) = \Phi^{(2)}(s_*) = 0$ (the second derivative is taken to be one-sided if s_* is one of the finitely many points of discontinuity of $\Phi^{(2)}$). So

$$\frac{d^2}{dt^2} \langle \tilde{\mu}, \tilde{\mu} \rangle_{x(s_*)} = 2 \frac{d}{dt} \langle \nabla_t \tilde{\mu}, \tilde{\mu} \rangle_{x(s_*)} = 2 \langle \nabla_t^2 \tilde{\mu}, \tilde{\mu} \rangle_{x(s_*)} + 2 \langle \nabla_t \tilde{\mu}, \nabla_t \tilde{\mu} \rangle_{x(s_*)} = 0.$$

So $\tilde{\mu}(s_*) = (\nabla_t \tilde{\mu})(s_*) = \mathbf{0}$. Since L is a second order linear differential operator, $\tilde{\mu}$ is identically $\mathbf{0}$, by Lemma 6. This contradiction of Lemma 5 proves Lemma 7.

Since $\tilde{\mu}$ is piecewise- C^2 , so is $\nabla_t x^{(1)}$ on any open interval $I \subset [t_0, t_1]$ where Φ is nonzero. So $x|I$ is piecewise- C^4 . So $\tilde{\mu}|I$ is also piecewise- C^4 , and so on. In fact $x|I$ and $\tilde{\mu}|I$ are C^∞ . So $\Phi^{(2)}$ is continuous except where $\tilde{\mu}$ is $\mathbf{0}$. So x and $\tilde{\mu}$ are C^∞ except where Φ is 0.

Theorem 1 is now proved for the special case where M is open in \mathbb{R}^m . As argued at the end of §3, this completes the proof of Theorem 1 in general.

7. Spheres

Take M to be the unit m -sphere S^m in E^{m+1} . The Levi-Civita covariant derivative $\nabla_Y Z$ of a vector field Z on S^m in the direction of $Y \in TS_y^m = y^\perp$ is given by

$$\nabla_Y Z = dZ_y(Y) + \langle Y, Z \rangle y$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, Z is treated as a map from S^m to E^{m+1} , and $y \in S^m$. Accordingly, the Riemannian curvature R on S^m is given by

$$R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y \quad \text{and, for a field } X \text{ along } x : [t_0, t_1] \rightarrow S^m,$$

$$L(X) = X^{(2)} + \langle x^{(1)}, x^{(1)} \rangle X + (\langle x^{(2)}, X \rangle + 2\langle x^{(1)}, X^{(1)} \rangle)x.$$

If x is optimal then, by Theorem 1, for some $z \in \mathbb{R}$ and some field X with a discrete set of zeroes, we have

$$x^{(2)} = zX/\|X\| - \langle x^{(1)}, x^{(1)} \rangle x \tag{15}$$

$$X^{(2)} = -\langle x^{(1)}, x^{(1)} \rangle X - (\langle x^{(2)}, X \rangle + 2\langle x^{(1)}, X^{(1)} \rangle)x \tag{16}$$

wherever $X(t) \neq \mathbf{0}$.

Example 4. Take $m = 2$, $z = 1.2$, and

$$x(0) = (1, 0, 0), \quad x^{(1)}(0) = (0, 1, 0), \quad X(0) = (0, 1, 200), \quad X^{(1)}(0) = (-1, 2, 1).$$

The solution $x : [0, 8] \rightarrow S^2$ is shown, viewed from $(2, 2, -1)$, in Figure 1, beginning at the left, and ending at $x(8) \approx (-0.433207, 0.898726, 0.0679917)$.

8. Bi-Invariant Lie Groups

Take M to be a Lie group with a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$, as in [16] §21. The (left) Lie-reduction of a field Y along $x : [t_0, t_1] \rightarrow G$ is the curve Y_L in the Lie algebra \mathcal{G} of G given by

$$Y_L(t) := dL_{x(t)}^{-1} Y_{x(t)}$$

where $L(g) : G \rightarrow G$ denotes left-multiplication by $g \in G$. Given x_0, x_1, v_0, v_1 , and an $F_{x_0, x_1, v_0, v_1, t_0, t_1}$ -optimal curve $x : [t_0, t_1] \rightarrow G$, denote the left Lie reduction of $x^{(1)}$ by V . By Theorem 1, $z = \|V^{(1)}(t)\|$ is constant, and $X_L = \phi V^{(1)}$ where $X := \phi \nabla_t x^{(1)}$.

Since X_L is a multiple of $V^{(1)}$, comparison of (2) with Lemmas 1, 2 of [18] gives

$$\tilde{X}_L^{(2)} = [\tilde{X}_L^{(1)}, V] \tag{17}$$

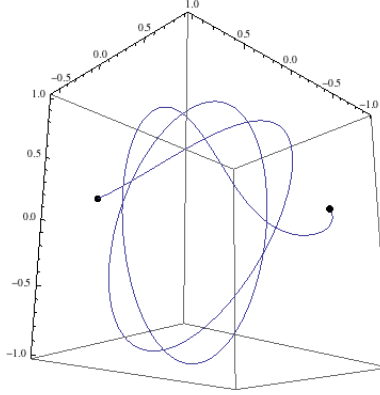


Figure 1: $x : [0, 8] \rightarrow S^2$ in Example 4

namely $X_L^{(1)}$ defines a Lax constraint [20] on V . So by Theorem 2.1 of [21], if G is semisimple, x is generically obtainable by quadrature from $X_L^{(1)} = \phi^{(1)}V^{(1)} + \phi V^{(2)}$ and V . Also $c := \|X_L^{(1)}\|^2 = z^2\phi^{(1)}(t)^2 + \phi(t)^2\|V^{(2)}(t)\|^2$ is constant, by (17). Integrating (17),

$$X_L^{(1)} = [X_L, V] + C \quad (18)$$

where $C \in \mathcal{G}$ is constant. Taking inner products of both sides of (18) with $V^{(1)}$,

$$z\phi^{(1)} = \langle C, V^{(1)} \rangle \implies z\phi(t) = \langle C, V(t) \rangle + a$$

where a is constant. Taking inner products of both sides of (18) with V ,

$$\langle X_L^{(1)}, V \rangle = \langle C, V \rangle = z\phi - a.$$

By analogy with Riemannian cubics [18], the optimal curve x is called *null* when $C = \mathbf{0}$. Then, since X_L defines another Lax constraint on V , $\|X_L\|^2 = z^2\phi(t)^2$ is constant. So ϕ is constant unless x is geodesic. Consequently, by equation (2) in Theorem 1, we have the

Corollary 1. *For M a bi-invariant Lie group, $x \in F_{x_0, x_1, v_0, v_1, t_0, t_1}$ is a null optimal curve if and only if x is a null Riemannian cubic.*

Null Riemannian cubics are studied in [18], [20], [22] for $G = SO(3)$, but most optimal curves are non-null and have different geometry, as seen from their reductions.

Example 5. *Take $G = SO(3)$, with the bi-invariant Riemannian metric for which*

$$\text{ad} : E^3 \rightarrow \mathfrak{so}(3)$$

is an isometry, where Euclidean 3-space E^3 is viewed as a Lie algebra with respect to the cross-product \times . Then the Lie reduction of an optimal curve $x^{(1)}$ can be identified with a curve V in E^3 , satisfying

$$V^{(1)}(t) = zW/\|W\| \quad (19)$$

$$W^{(1)}(t) = W \times V + C \quad (20)$$

where $z \in \mathbb{R}$, $C \in E^3$, $\| \cdot \|$ is the Euclidean norm $\| \cdot \|_E$, and $W(t) := X_L(t) \in E^3$. As noted previously, there are conserved quantities

$$c = \|W^{(1)}\|^2, \quad a = z\|W\| - \langle C, V \rangle = z\|W\| - \langle W^{(1)}, V \rangle.$$

Taking $z = 1.2$, $C = -(2, 1, 0)$, $V(0) = (1, 2, 3)$, $W(0) = (-1, -4, 6)$, a numerical solution of (19), (20) for $V : [0, 700] \rightarrow E^3$ is shown in Figure 2 below. The inner dot is at $V(0)$ and the outer is at the other endpoint $V(700) \approx (2.36765, 4.69752, 8.40276)$, and the image of V is viewed from $(10, 8, 0)$. The curve is nonplanar and seems to be approximately periodic. Similarly, $\phi = \|W\| : [0, 700] \rightarrow \mathbb{R}$ seems to be approximately

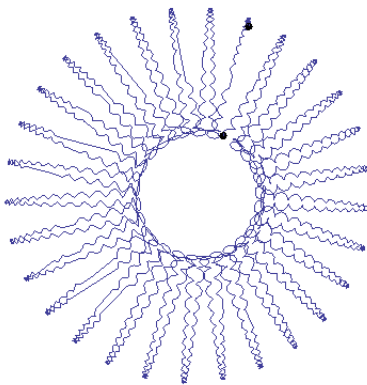


Figure 2: Lie Reduction of $x^{(1)} : [0, 700] \rightarrow SO(3)$ in Example 5

periodic, and is apparently everywhere nonzero. The graph over the interval $[0, 55]$ is shown in Figure 3. If C is replaced by $(2, 1, 0)$, the reduction V of $x^{(1)}$ appears to be unbounded. The image of $V|_{[0, 5]}$ shown in Figure 4 resembles that of a null Lie quadratic, spiralling curve inwards towards an asymptotic line. However x is not a null Riemannian cubic, because $C \neq \mathbf{0}$. The lower dot corresponds to $V(0)$ and the upper dot is at $V(5) \approx (1.77133, 4.50895, 7.05963)$.

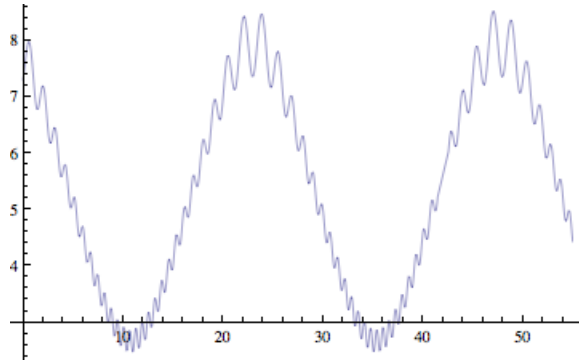


Figure 3: ϕ in Example 5

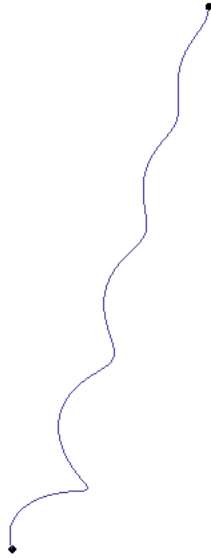


Figure 4: A nonperiodic reduction of a non-null optimal curve in Example 5

9. Two Points and One Velocity

So far, for feasible curves $x : [t_0, t_1] \rightarrow M$, $x(t_i)$ and $x^{(1)}(t_i)$ have been prescribed for $i = 0, 1$. With no conditions on the $x^{(1)}(t_i)$, J would be minimised by a geodesic. Consider next what happens when the condition on $x^{(1)}(t_1)$ is lifted.

Definition 3. Let $x_0, x_1 \in M$ and $v_0 \in TM_{x_0}$ be given. Let $F_{x_0, x_1, v_0, t_0, t_1}$ be the set of piecewise- C^2 curves for which $x(t_i) = x_i$ for $i = 0, 1$ and $x^{(1)}(t_0) = v_0$. Defining $J : F_{x_0, x_1, v_0, t_0, t_1} \rightarrow \mathbb{R}$ in the same way as before, a minimiser $x \in F_{x_0, x_1, v_0, t_0, t_1}$ of J is said to be $F_{x_0, x_1, v_0, t_0, t_1}$ -optimal.

Theorem 2. Let x be $F_{x_0, x_1, v_0, t_0, t_1}$ -optimal. Then $\|\nabla_t x^{(1)}\|_{x(t)}$ is constant. For some piecewise- C^2 function $\Phi : [t_0, t_1] \rightarrow [0, \infty)$ with $\Phi^{-1}(0)$ a discrete set containing t_1 , we have

$$L(\phi \nabla_t x^{(1)}) = \mathbf{0} \quad (21)$$

where $\phi(t) := \sqrt{\Phi(t)}$. Moreover x and ϕ are C^∞ except possibly where $\Phi(t) = 0$.

The change in the conclusion from Theorem 1 is that $\Phi(t_1) = 0$. We omit the changes of detail needed for the proof of Theorem 2. Considering track-inverses, a similar result holds when $x_0, x_1 \in M$ and $v_1 \in TM_{x_1}$ are given, and feasibility means $x(t_i) = x_i$ for $i = 0, 1$ with $x^{(1)}(t_1) = v_1$.

10. Many Points and No Velocities

Another kind of feasibility occurs when, for $n \geq 3$, we are given real numbers $t_0 < t_1 < \dots < t_j < \dots < t_n$ and points $x_0, x_1, \dots, x_j, \dots, x_n \in M$. Denote by $F_{x_0, x_1, \dots, x_n, t_0, t_1, \dots, t_n}$ the set of piecewise- C^2 curves $x : [t_0, t_n] \rightarrow M$ satisfying $x(t_i) = x_i$ for $i = 0, 1, \dots, n$. Define $J : F_{x_0, x_1, \dots, x_n, t_0, t_1, \dots, t_n} \rightarrow \mathbb{R}$ by

$$J(x) := \max\{\|\nabla_t x^{(1)}\|_{x(t)} : t_0 \leq t \leq t_n\}.$$

Then a minimiser $x \in F_{x_0, x_1, \dots, x_n, t_0, t_1, \dots, t_n}$ of J is called $F_{x_0, x_1, \dots, x_n, t_0, t_1, \dots, t_n}$ -optimal. On considering optimality of track-summands, we find from Theorems 1, 2,

Corollary 2. Let x be $F_{x_0, x_1, \dots, x_n, t_0, t_1, \dots, t_n}$ -optimal. Then, for some $j = 1, 2, \dots, n$ and all $t \in [t_{j-1}, t_j]$, $\|\nabla_t x^{(1)}\|_{x(t)}$ is constant. Moreover,

- for some piecewise- C^2 function $\Phi : [t_{j-1}, t_j] \rightarrow [0, \infty)$ with $\Phi^{-1}(0)$ discrete, we have $L(\phi \nabla_t x^{(1)}) = \mathbf{0}$ for $t \in [t_{j-1}, t_j]$, where $\phi(t) := \sqrt{\Phi(t)}$,

- x and ϕ are C^∞ except possibly where $\Phi(t) = 0$,
- if $j = 1$ then $\Phi(t_0) = 0$,
- if $j = n$ then $\Phi(t_n) = 0$.

Proof of Corollary: Suppose there is no such j . Then, by Theorems 1, 2, for every $j = 1, 2, \dots, n$, $x|_{[t_{j-1}, t_j]}$ is suboptimal with respect to the restriction of J_∞ to $F_{x(t_{j-1}), x(t_j), x^{(1)}(t_{j-1}), x^{(1)}(t_j)}$. This contradicts optimality of x .

Example 6. For a complete Riemannian manifold M , choose $x_0, x_1, x_2 \in M$ not lying on a geodesic. Choose real numbers $t_0 < t_1 < t_2$ and let x be $F_{x_0, x_1, x_2, t_0, t_1, t_2}$ -optimal. Set $\varepsilon := J_\infty(x)$. Then $\varepsilon > 0$.

For some $t_3 > t_2$ let $y : [t_2, t_3] \rightarrow M$ be a geodesic and set $x_3 := y(t_3)$. Let $\tilde{y} : [t_2, t_3] \rightarrow M$ be any piecewise- C^2 curve with $\tilde{y}^{(1)}(t_2) = x^{(1)}(t_2)$, $\tilde{y}(t_3) = x_3$ and $\|\nabla_t \tilde{y}^{(1)}\|_{\tilde{y}(t)} < \varepsilon$ for all $t \in (t_2, t_3)$. For instance \tilde{y} could be chosen as a perturbation of y , or indeed y itself.

Then the track-sum $\bar{x} : [t_0, t_3] \rightarrow M$ of x and \tilde{y} lies in $F_{x_0, x_1, x_2, x_3, t_0, t_1, t_2, t_3}$. Moreover $J_\infty(\bar{x}) = J_\infty(x)$, and \bar{x} is easily seen to be $F_{x_0, x_1, x_2, x_3, t_0, t_1, t_2, t_3}$ -optimal. Yet, since there are continuously many possibilities for \bar{x} , the conditions of Corollary 2 do not apply for $j = 3$.

11. Conclusion

This paper proves necessary conditions for minimum L^∞ acceleration curves in Riemannian manifolds M , analogous to the conditions in [17] for Riemannian cubics. Results of this kind were previously known only when the manifold M was flat. In the present paper the necessary conditions are examined in detail when M is a sphere or a bi-invariant Lie group. Examples are given in the case where M is S^2 or $SO(3)$, raising questions about the asymptotics and symmetry of optimal curves.

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